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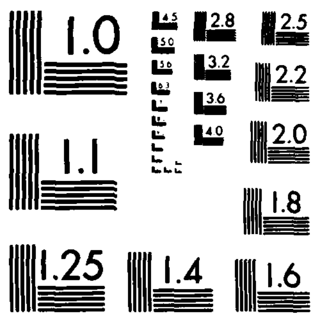
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NAVAL UNDERWATER SYSTEMS CENTER
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Technical Memorandum

DETECTION OF RANDOMLY OCCURRING SIGNALS USING SPECTRA AND FREQUENCY DOMAIN KURTOSIS ESTIMATES

Date: 23 March 1984

Prepared by: Roger F. Dwyer
Roger F. Dwyer

Approved for public release; distribution unlimited

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ABSTRACT

Several detection statistics are compared in the frequency domain based on the asymptotic probability of detection criterion. ~~They~~ These include, second-order, fourth-order, and two forms of kurtosis estimates. The results show that for randomly occurring signals or non-Gaussian signals, the fourth-order and kurtosis estimates can have higher asymptotic probability of detection levels compared with second-order estimates. But, only for the kurtosis estimates do the results seem significant. Moreover, if a second-order estimate of the noise is available to normalize a fourth-order estimate of signal and noise, the resultant modified kurtosis estimate has higher asymptotic probability of detection levels even for Gaussian signals. This result only holds when there is a significant positive covariance between the numerator and the normalizing noise sample in the denominator. On the other hand, if an independent noise sample is used to normalize a second-order or fourth-order estimate the overall performance based on the asymptotic probability of detection will be degraded compared with the unnormalized second-order or fourth-order estimates, respectively. This result could impact current sonar processing methods.

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I. INTRODUCTION

In this paper the performance of several detection statistics are compared based on the asymptotic probability of detection criterion. The likelihood ratio (LR) is considered a general method for deriving optimum detection statistics or receiver structures. However, the particular form of the likelihood ratio may not be practical to implement, or, on the other hand, the circumstances under which the LR is designed may change. In these cases it may be of interest to consider detection statistics other than second-order which may have better performance characteristics for non-Gaussian signals. The emphasis is on detection here, but, kurtosis estimates can also distinguish between Gaussian and non-Gaussian signals. In applications this is an important property.

To motivate our analyses to follow the form of the LR will be derived under conditions which can be supported by physical evidence. But, as has been suggested already, the actual LR statistic for any particular condition will not be analyzed in the paper. Rather, we will be interested in a statistic which may be applied to a wide range of situations where non-Gaussian signal may arise.

Let there be available a set of N independent and identically distributed (i.i.d.) samples, as depicted in figure 1. The joint probability density function (pdf) for these samples may be written as

$$f(x_1, x_2, . . ., x_n) = \prod_{i=1}^{N} f_1(x_i) \qquad (1)$$

Equation (1) will represent the noise only hypothesis and will be denoted by H_0.

Now, if in the set of N independent samples some belong to another probability distribution, say $F_2(x)$, representing the signal and noise and identified by H_1, as depicted in figure (2), then the likelihood ratio for the N samples would be given by

$$LR = g(x_1, x_2, \ldots, x_n)/f(x_1, x_2, \ldots, x_n) \tag{2}$$

where $g(x_1, x_2, \ldots, x_n)$ represents the joint pdf for signal and noise. Before giving the form of equation (2), consider the case when only one sample belongs to $F_2(x)$ out of a possible set N.

In this case, from the law of total probability the joint pdf of signal and noise is given by for independent samples

$$\begin{aligned} g(x_1, x_2, \ldots, x_n) = {} & P_1\, f_2(x_1) \prod_{i=2}^{N} f_1(x_i) \\ & + P_2\, f_1(x_1) f_2(x_2) \prod_{i=3}^{N} f_1(x_i) \\ & + \\ & \vdots \\ & + P_K \prod_{i=1}^{K-1} f_1(x_i)\, f_2(x_K) \prod_{j=k+1}^{N} f_1(x_j) \\ & + \\ & \vdots \\ & + P_N\, f_2(x_N) \prod_{i=1}^{N-1} f_1(x_i)\ , \end{aligned} \tag{3}$$

where, P_k, $k = 1, 2, \ldots, N$, is the probability that the k-th sample belongs to $F_2(x_k)$. By forming the LR between equation (3) and equation (1) we obtain

$$LR = \sum_{K=1}^{N} P_k[f_2(x_k)/f_1(x_k)] \quad . \tag{4}$$

Therefore, the LR of equation (4) is formed by averaging the individual sample likelihood ratios over all possible sample positions of the signal. If it is known when the signal, belonging to $F_2(x)$, occurs, say the k-th, then $P_k = 1$ and, therefore, $LR = f_2(x_k)/f_1(x_k)$. Thresholding is sometimes used to establish when the sample belonging to $F_2(x)$, i.e., the signal, occurs. But this method will not be considered here since it appears to presuppose the solution before it can be derived from fundamental principles.

The likelihood ratio of the more general case of M samples belonging to $F_2(x)$ can be written as follows:

$$LR = \frac{M!(N-M!)}{N!} \sum_{K_1=1}^{N-(M-1)} \; \sum_{K_2=K_1+1}^{N-(M-2)} \cdots \sum_{K_{M-1}=K_{M-2}+1}^{N-1} \; \sum_{K_M=K_{M-1}+1}^{N} f_s(x_{K_1}, x_{K_2}, \ldots, x_{K_M}), \tag{5}$$

where, $f_s(x_{K_1}, x_{K_2}, \ldots, x_{K_M}) = f_2(x_{K_1})f_2(x_{K_2}) \ldots f_2(x_{K_M})/f_1(x_{K_1})\, f_1(x_{K_2}) \ldots f_1(x_{K_M})$,

and it was assumed that the probability of any particular configuration was equally likely, i.e., $P_{K_1 K_2 \ldots K_M} = M!(N-M)!/N!$. Therefore, the LR of equation (5) is formed for the N sampling positions by averaging over all possible combinations of the M samples containing the signal.

If M = 1 equation (5) reduces to equation (4) with $P_k = 1/N$. However, if M = N then equation (5) reduces to

$$LR = \prod_{i=1}^{N} [f_2(x_i)/f_1(x_i)] , \tag{6}$$

which is the form of the likelihood ratio treated by Marcum [1, p. 209].

These results can also be extended to include dependent samples. For example, for Markov dependence, equations (4) and (5) will be functions of conditional probability densities and the LR structure will be, therefore, more complicated.

From these cases it appears that the form of the LR depends upon what is known about the occurrence of the signal over the interval in addition to the usual requirements of knowledge of the pdf's. If we design our receiver based on equation (6) and the signal occurs only for a percentage of the N samples then we would expect the performance of the receiver to be degraded over what could be achieved under the proper likelihood ratio formulation. Therefore, the LR based on equation (6) may not lead to the optimum detection statistic for randomly occurring signals. This has been shown to be the case by Ferguson [2] where skew and kurtosis estimates were optimum statistics under suitably chosen conditions. Since in practice the previously derived LR structures may be too costly to implement this paper will compare various forms of the kurtosis estimate with a second-order estimate. These statistics will be simpler to implement but also may lead to improved performance.

II. ASYMPTOTIC PROBABILITY OF DETECTION

The introduction considered signals which could occur randomly over a sample set N in a one-dimensional space, or only over discrete temporal locations. Often in applications it is desired to operate on signals in the frequency domain. This domain would allow signals to occur randomly in two-dimensions, vis., frequency and temporal locations. An application to the ice-induced signals was discussed by Dwyer in reference [3]. This idea can also be extended to three-dimensions by taking into account the spatial locations. Here we formulate the problem in the frequency domain but the basic properties would be applicable in any dimension. The higher dimensions, however, would offer more opportunities for the signal to occur randomly.

Let $x(i,q) = x\{[i+(q-1)M]h\}$, $i=0,1,\ldots,M-1$, $q = 1,2,\ldots,n$,

represent real discrete data with h representing the interval between consecutive samples. The discrete Fourier transform (DFT) is defined as follows:

$$x(q,F_p) = \sqrt{h/M} \sum_{i=0}^{M-1} W_i x(i,q) \exp(-jF_p i), \qquad (7)$$

where, $j = \sqrt{-1}$, $F_p = 2\pi f_p h$ is the p-th radian frequency component, $p = 0,1,\ldots,M-1$ and $f_p = P/Mh$ Hz. For simplicity, let the window weights equal one, i.e., $W_i = 1$, $i=0,1,\ldots,M-1$, and $h = 1$.

The time domain data will be represented as an additive sum of signal and noise of the form,

$$x(i,q) = N(i,q) + m(i,q)\ S(i,q) \tag{8}$$

where the noise N(i,q) and signal S(i,q) are zero-mean Gaussian process. The function m(i,q) modulates the signal in such a way that the frequency domain representation will be a *non-Gaussian process* [3].

Nuttall [4] considered a signal that was modulated by a random constant and, therefore, was not a function of time. This kind of modulation as was found by Nuttall would not lead to a non-Gaussian process based on the model and, therefore, the LR of equation (6) would be appropiate. In contrast, the modulation function of this paper is a random function of time over the detection interval and leads to non-Gaussian process. This model is supported by real data measurements as discussed in reference [3].

The power spectrum density (PSD) which represents a second-order estimate is defined as [5,6]

$$P(F_p) = (1/n) \sum_{q=1}^{n} x(q,F_p)x^*(q,F_p), \tag{9}$$

where the asterisk represents complex conjugate. The asymptotic variance of power spectra and complex cross spectra were discussed in reference [7]. The parameter n is sometimes called the degrees of freedom of the estimate.

Let the frequency domain kurtosis (FDK) estimate [3] be defined for real and imaginary parts of equation (7) separately. We can therefore discuss the theoretical properties of the FDK for its real part only. The imaginery part has, identical properties. In applications both real and imaginary parts are estimated, since both contain information. In addition, the real and imaginary parts of the FDK estimate can be combined to form one statistic, say, for example, the magnitude. But these considerations are not expected to change the conclusions to follow in any significant way. Although, a performance improvement based on the magnitude of the FDK estimates over the real or imaginary part separately would be expected compared with the PSD. An example for the magnitude of the FDK estimates will be given later.

The FDK estimate for the real part of equation (7) is defined assuming that $x(q,F_p)$ is a zero-mean process as

$$K(F_p) = (1/n) \sum_{q=1}^{n} [x(q, F_p)]^4 / \left\{(1/n) \sum_{q=1}^{n} [x(q,F_p)]^2\right\}^2 . \tag{10}$$

If $x(q,F_p)$ is not a zero-mean process then the mean would be estimated from the data and subtracted from $x(q,F_p)$. So, therefore, the mean can be accounted for in principle and need not concern us further.

The asymptotic probability of detection (APD) introduced in reference [8] will be derived for three specific cases of equation (10) and compared with the APD results of equation (9). In case 1, the numerator and denominator of equation (10) will contain signal and noise as shown in the data model of equation (8). This was the form of the frequency domain kurtosis estimate

used in the real data analysis of reference [9]. For case 2, the denominator will contain a noise only estimate. The results of this case are of theoretical interest. However, some comments concerning implementing this case will be discussed later. In the last case the denominator of equation (10) will be set equal to one. Thus, case 3 represents a fourth-order moment estimate.

Since only the real part will be treated theoretically, we shall write equation (7) as follows:

$$x(q,F_p) = (1/M)^{1/2} \sum_{i=0}^{M-1} N(i,q) \cos (F_p i) + m(q) \sum_{i=0}^{M-1} S(i,q) \cos (F_p i) , \qquad (11)$$

where m(i,q) was assumed to change slowly with i and therefore, could be w approximated as m(i,q) = m(q).

To simplify notation equation (11) will be expressed for a particular frequency as,

$$x(q) = N(q) + m(q)S(q). \qquad (12)$$

where N(q), m(q), and S(q) are mutually independent. In addition, x(q) will be assumed a zero-mean process and statistically independent, i.e., $E[x(q_1)x(q_2)] = \delta(q_1-q_2)\ E[x(q_1)^2]$. Where $\delta(q_1-q_2)$ is the Kronecker delta function. The independent assumption is needed in order to evaluate the variance of equations (9) and (10). However, the time series x(i,q) with respect to i may be considered dependent. This point will be discussed more fully later.

The modulating term m(q) will be modeled as a Bernoulli time series [3] defined as follows:

$$m(q) = \begin{cases} 1 & ; \quad P_r\{m(q) = 1\} = L \\ 0 & ; \quad P_r\{m(q) = 0\} = 1 - L. \end{cases}$$

Therefore, all moments of m(q) are given by $E[m(q)^r] = L$, $r = 1, 2, \ldots$. Thus, m(q) modulates the signal, in the frequency domain at a specific frequency, by turning it on or off over the detection interval. The probability (L) of the signal being on will be a parameter of the performance results.

The asymptotic ($n \to \infty$) behavior of equation (9) and (10) can be obtained from Cramer's convergence proofs [10].

For a function $F(x_1, x_{2z})$, z = 1, 2, 3, of two central moments x_1, x_{2z}, from a one-dimensional sample corresponding to equation (9) and (10) Cramer has shown that as $n \to \infty$, $F \to N\{E(F), var(F)\}$, where $N\{\ ,\ \}$ represents a normal process with mean and variance given by, respectively,

$$E[F] = F[E(x_1), E(x_{2z})] + O(1/n),$$

$$VAR[F] = VAR(x_1)F_1^2 + VAR(x_{2z})\ F_2^2$$

$$+ 2F_1F_2COV(x_1\ x_{21}) + O(1/n^{3/2}).$$

The parameters F_1 and F_2 are partial derivatives of F, i.e.,

$F_1 = (\partial F/\partial x_1)\Big|_{x_1 = E(x_1)}$, and, $F_2 = (\partial F/\partial x_{2z})\Big|_{x_{2z} = E(x_{2z})}$

In the following analysis we shall replace F by P for the PSD and replace F by K for the FDK when evaluating their respective means and variances.

Since the asymptotic process is Gaussian, the asymptotic probability of detection (APD) can be written as follows:

$$APD = 1 - \Phi\left\{\Phi^{-1}(1-\alpha)\ \sigma_0(F)/\sigma_1(F) + [E_0(F) - E_1(F)]/\sigma_1(F)\right\}\ , \qquad (13)$$

where, $E_i(\ )$, $\sigma_i(\ )$ are, respectively, the mean and standard deviation of the noise only process when $i = 0$, or the signal and noise process when $i = 1$. $\Phi\{\ \}$ is the standard error function and α represents the desired false alarm probability.

III. THEORETICAL DETECTION PERFORMANCE RESULTS

A. Power Spectrum Density

The power spectrum density (PSD) estimate for the real part of equation (9) is given by

$$P(F_p) = F(x_1,1) = (1/n) \sum_{q=1}^{n} x(q)^2 \quad . \qquad (14)$$

The required components of the APD are

$E[P(F_p)] = N(1 + L\ SNR)$, $VAR\ [P(F_p)] = (1/n)\ 2N^2\ a_1$,

where, $N = E[N(q)^2]$, $S = E[S(q)^2]$, $SNR = S/N$, and

$a_1 = 1 + 2L\ SNR + [3L - L^2)/2]\ SNR^2$.

Recall that the variance of the periodogram, as well as equation (14), does not decrease with increasing transform size M [6]. We shall prove later that the variance of the FDK is also independent of M and like the PSD both variances decrease proportional to (1/n) if n .ꞩ sufficiently large.

By substituting the above parameters for the PSD into equation (13) we obtain

$$APD = 1 - \left\{\Phi[\Phi^{-1}(1 - \alpha) - L\ SNR\sqrt{n}/\sqrt{2}]/\sqrt{a_1}\right\}. \qquad (15)$$

Equation (15) will be evaluated for specific values of the parameters and compared with the results for the FDK of case 1 in the next section.

Another point should be mentioned concerning the PSD before proceeding. Later we shall discuss the performance of a fourth order estimate normalized by a noise only second-order estimate, which is designated as case 2. This procedure leads to higher APD values compared with all other cases including the PSD even for Gaussian processes. It is believed the reason for this depends on the noise only second-order estimate normalizing function. But, the PSD can also be normalized by a noise only second-order estimate. When this is done it appears that the PSD would have a larger APD for Gaussian proceses. However, the false alarm probability cannot be controlled at a satisfactory small level but is fixed at .5. A proof of this is given in Appendix A.

B. Frequency Domain Kurtosis, Case 1

From the results of the previous section the FDK can be expressed as follows:

$$k_z(F_p) = x_1/x_{2z} \ , \tag{16}$$

where,

$$x_1 = (1/n) \sum_{q=1}^{n} x(q)^4 \ ,$$

$$x_{2z} = [(1/n) \sum_{q=1}^{n} x(q)^2]^2 \ .$$

Therefore, for z = 1, case 1, the asymptotic expected value of equation (16) reduces to

$$\lim_{n \to \infty} E[k_1(F_p)] = E[x_1/x_{21}] = E[x(q)^4]/E[x(q)^2] \ .$$

Since all our results are asymptotic ($n \to \infty$) we shall not explicitly indicate the limiting process in the following expressions.

Using the data model of equation (12) the expected value of $k_1(F_p)$ can be put in the parmeterized form

$$E[k_1(F_p)] = 3 \ \ a_2/a_3^2,$$

where, $a_2 = 1 + 2L\ SNR + L\ SNR^2$, and $a_3 = 1 + L\ SNR$.

The derivaton of the variance of equation (16) requires evaluating several components. Some of them are tedious calculations. Therefore, the main points of the derivation will only be outlined here. The details of the derivation will be found in Appendix B.

Based on Cramer's [10] convergence proof the form of the variance of $k_1(F_p)$ is given by

$$\begin{aligned} VAR[k_1(F_p)] &= VAR\ (x_1)\ F_1^2 + VAR(x_{21})\ F_2^2 \\ &\quad + 2\ F_1\ F_2\ \mathrm{cov}(x_1\ x_{21})\ , \end{aligned}$$

where,

$$F_1 = (\partial\ k_1(F_p)/\partial x_1)\Big|_{x_1 = E(x_1)} = E[x(q)^2]^{-2}\ ,$$

$$F_2 = (\partial\ k_1(F_p)/\partial\ x_{21})\Big|_{x_{21} = E(x_{21})} = -E[x(q)^4]\ E[x(q)^2]^{-4}\ .$$

The other components are given by

$$VAR(x_1) = (1/n)\ VAR[x(q)^4]$$

$$VAR(x_{21}) = E\left\{[(1/n) \sum_{q=1}^{n} x(q)^2]^4\right\} - \left\{E\left\{[(1/n) \sum_{q=1}^{n} x(q)^2]^2\right\}\right\}^2$$

$$\mathrm{cov}[x_1\ x_{21}] = E(x_1\ x_{21}) - E[x_1]\ E[x_{21}]$$

$$= E\left\{(1/n^3) \sum_{q_1=1}^{n} \sum_{q_2=1}^{n} \sum_{q_3=1}^{n} x(q_1)^4\ x(q_2)^2\ x(q_3)^2\right\}$$

$$-E[x(q)^4]\; E\left\{[(1/n)\sum_{q=1}^{n} x(q)^2]^2\right\} \quad .$$

By expanding the components further we obtain,

$$\begin{aligned} VAR[k_1(F_p)] = (1/n)\Big\{ & VAR[x(q)^4]/E[x(q)^2]^4 \\ & + 4\left\{E[x(q)^4]\; E[x(q)^2]^2 - E[x(q)^2]^4\right\} E[x(q)^4]^2/E[x(q)^2]^8 \\ & - 4\left\{E[x(q)^6]\; E[x(q)^2] - E[x(q)^4]\; E[x(q)^2]^2\right\}/E[x(q)^2]^6 \Big\} . \end{aligned}$$

Substituting the data model of equation (12) into the above expressions the desired parameterized form for the variance of case 1 reduces to the following.

$$VAR[k_1(F_p)] = (1/n)\; Q_1 \; , \qquad (17)$$

where,

$$Q_1 = 96\; D_1 + 72\; D_2 - 144\; D_3$$

$$D_1 = a_4/a_2^2$$

$$\begin{aligned} a_4 = {} & 1 + 4L\; SNR + [612L - 36L^2)/96]\; SNR^2 + [(420L - 36L^2)/96] SNR^3 \\ & + [(10SL - 9L^2)/96]\; SNR^4 \end{aligned}$$

$$D_2 = (a_2^4/a_3^4)\; (1/8)\; [18 - 10\; a_3\; a_5/a_2]$$

$$a_5 = 1 + 2L\; SNR + [4L^2 + 6L)/10]\; SNR^2$$

$$D_3 = (a_2^3)a_3^3)(1/12)\ [15\ a_6(1 + L\ SNR)/a_6^2 - 3\ a_3/a_2]$$

$$a_6 = 1 + 3L\ SNR + 3L\ SNR^2 + L\ SNR^3.$$

Under the noise only case (SNR = 0) or when L equals 0 or 1 the variance of case 1 asymptotically approaches,

$$VAR[k_1(F_p)/H_0] = 24/n \tag{18}$$

This is the result obtained for a Gaussian process by Pearson [11].

Now we are in a position to express the APD for the FDK of case 1. The result is given by

$$APD = 1 - \Phi\{[\sqrt{24}\,\Phi^{-1}\ (1 - \alpha) + 3(1 - a_2/a_3)\sqrt{n}]/Q_1^{1/2}\}. \tag{19}$$

For L equal to 0 or 1 the APD of equation (19) equals the false alarm probability so detection is not possible under these conditions. Detection is also not possible for the PSD when L equals zero. But unlike the FDK of case 1 the probability of detection for the PSD is maximum when L equals one. The FDK of case 2 will also have this property. This means that $k_1(F_p)$ as a detection statistic is only sensitive to non-Gaussian ($0 < L < 1$) signals.

An example will demonstrate these results more clearly. Figure 3 represents a comparison of the PSD and FDK of case 1 as a function of the probability of occurrence (L) of the signal for several SNR values. The figure shows that the FDK can have a higher probability of detection

depending on SNR for small values of L only. As L increases the probability of detection for the PSD increases with L and will surpass the probability of detection for the FDK of case 1. As L approaches one the probability of detection for the FDK of case 1 decreases and approaches the false alarm probability (α).

A real data example will put these results in perspective. The following example has been explained in reference [3]. It is from Arctic under-ice environmental data. Figure 4 compares the power spectrum density estimate and the real part of the frequency domain kurtosis estimate for a segment of Arctic data. Both estimates are averaged over many consecutive FFT estimates. We shall concentrate on the frequency with the highest kurtosis estimate and a corresponding small PSD estimate on the figure. From the theoretical results we know that the signal causing the high kurtosis level only occurs a small percentage of time and when it occurs it probably has a high SNR level. This conclusion could not be reached from the PSD estimate alone. Therefore, the FDK estimate of case 1 contributes additional information. So, the usefulness of the FDK estimate of case 1 cannot be determined from its probability of detection performance level alone. But must be evaluated in terms of contributing additional information which the PSD estimate is insensitive to.

For the next two cases of the FDK we will explore ways of improving its probability of detection. But, on the other hand, this procedure may sacrifice the information content of the FDK estimate.

C. Frequency Domain Kurtosis, Case 2

For this case the denominator of equation (10) contains a noise only second-order estimate. The FDK estimate of case 2 follows from equation (16) as

$$k_2(F_p) = x_1/x_{22}. \tag{20}$$

The derivations of the following expressions are similar to the derivations for the FDK of case 1. Therefore, only the main points of the derivation will be explicitly stated in the following.

For sufficiently large n, the expected value of equation (20) is given by

$$E[k_2(F_p)] = E[x(q)^4/H_1]/ E[x(q)^2/H_0]^2 \tag{21}$$

Where we now signify which hypothesis the expected value is conditioned on. By substituting the data model of equation (2) into equation (21) we obtain the parameterized result,

$$E[k_2(F_p)] = 3\ a_2 \tag{22}$$

The variance is given by

$$\begin{aligned} VAR[k_2(F_p)] = (1/n)\Big\{ & VAR[x(q)^4/H_1]/E[x(q)^2/H_0]^2 \\ & + 4\ E[x(q)^4/H_1]^2 \{ E[x(q)^4/H_0]\ E[x(q)^2/H_0]^2 - E[x(q)^2/H_0]^4 \}/E[x(q)^2/H_0]^8 \end{aligned}$$

$$-2\left\{E[x(q)^4/H_1]\ /\ E[x(q)^2/H_0]^6\right\}\Bigg\{$$

$$E[(1/n)\sum_{q=1}^{n}[x(q)^4/H_1]\ [(1/n)\sum_{q=1}^{n}[x(q)^2/H_0]]^2$$

$$-E[(1/n)\sum_{q=1}^{n}x(q)^4/H_1]\ E[(1/n)\sum_{q=1}^{n}x(q)^2/H_0]^2\Bigg\}\Bigg\} \quad . \tag{23}$$

The last term of equation (23) represents the covariance of x_1 and x_{22}, i.e, cov[$x_1 x_{22}$]. If the noise only estimate x_{22} was statistically independent of x_1 then the covariance would equal zero. This may happen if the noise estimate is from an adjacent frequency location or from another interval of time. This would change the APD performance results considerably.

Employing the data model we obtain the parameterized result

$$\mathrm{VAR}[k_2(F_p)] = (1/n)\ Q_2 \tag{24}$$

where $Q_2 = 96\ a_4 + 72\ a_2^2 - 144\ a_2\ a_3$.

If the covariance is zero the third term of Q_2 would be zero giving a higher variance for $k_2[k_2(F_p)]$. Now the role of the normalizing component, x_{22}, is clear. If x_{22} is correlated with the noise component of x_1 then the effect is to reduce the variance of $k_2(F_p)$.

The APD of case 2 also shows this relationship.

$$\mathrm{APD} = 1 - \Phi\left\{[\sqrt{24}\ \Phi^{-1}\ (1 - \alpha) - 3\ a_7\sqrt{n}]/Q_2^{1/2}\right\} \quad , \tag{25}$$

where $a_7 = 2L\ SNR + L\ SNR^2$.

Therefore, if Q_2 is minimized the probability of detection can be improved. The APD is maximum for the FDK of case 2. In practice a partially correlated noise sample might be available to improve the detection performance of the FDK estimate. An example will be given later to demonstrate the idea. But before given examples of the results of case 3 will be given.

D. Frequency Domain Kurtosis, Case 3

Recall, for this case the denominator of equation (10) is set equal to one. The derivations of the mean and variance of case 3 are similar to the previous cases so we will only present the results.

The expected value and variance for the FDK of case 3 are given by

$$E[K_3(F_p)] = 3\ N^2\ a_2 \quad ,$$

$$VAR[K_3(F_p)] = (1/n)\left(96\ N^4\ a_4\right)$$

The APD follows as

$$APD = 1 - \Phi\left\{[\Phi^{-1}(1 - \alpha) - 3\ a_7\ \sqrt{n}/\sqrt{96}]/\sqrt{a_4}\right\} \ . \qquad (26)$$

At this point it is appropriate to consider the performance of a V-th order power law using the data model of equation (12). Recall that both N(q) and S(q) are assumed to be statistical independent and zero-mean Gaussian

processes. Here we shall utilize one degree of freedom, so the results are not asymptotic.

The V-th order power law is defined as follows

$$y = x^{v} \tag{27}$$

where for v even, $v = 2k$, $k = 1,2,\ldots,$, and for v odd, $v = 2k-1$, $k=1,2,\ldots$.

When v equals two the results will apply to equation (14) and when v equals four they will apply to equation (16), where $x_{23} = 1$.

Case (I) V EVEN

By rewriting equation (27) in the following way

$$y \exp(jn2\pi) = x^{v},\ n=0,\pm1,\pm2,\ldots,$$

the solutions for x can be obtained. For $y > 0$

$$x = y^{1/v} \exp(jn2\pi/v).$$

By examining the possible solutions we find that there are only two real solutions for x, viz.,

$$x_1 = y^{1/v} \text{ and } x_2 = -y^{1/v}.$$

There are no real solutions for x when $y < 0$. Therefore, the probability density function of y is given by

$$f_Y(y) = \lfloor vy^{(v-1)/v} \rfloor^{-1} \lfloor f_X(y^{1/v}) + f_X(-y^{1/v}) \rfloor, \; y > 0$$

$$f_Y(y) = 0 \quad , \; y < 0.$$

From the characteristic function of equation (12) the input probability density function can be shown to be given by the following mixture density

$$f_X(x) = (1-L)\, f_1(x) + Lf_2(x), \tag{28}$$

where, $f_1(x)$ is $N\left\{0,\ \sigma_n\right\}$, and $f_2(x)$ is $N\left\{0,\ \sigma\right\}$, $\sigma = \lfloor \sigma_n^2 + \sigma_s^2 \rfloor^{1/2}$.

Therefore, since a Gaussian process is symmetric about its mean value we obtain

$$f_Y(y) = 2\lfloor vy^{(v-1)/v} \rfloor^{-1} \left\{(1-L)\, f_1(y^{1/v}) + Lf_2\,(y^{1/v})\right\} , \; y > 0$$

and

$$F_Y(y) = 2\left\{(1-L)\lfloor\Phi(y^{1/v}/\sigma_n) - 1/2\rfloor + L\lfloor\Phi(y^{1/v}/\sigma) - 1/2\rfloor\right\}, \; y > 0$$

where, $\Phi(\ )$ is the Gaussian error function.

Since we are interested in the performance properties with one degree of freedom, we shall let L=1, i.e., the signal occurs with probability one on this sample. The objective is to find the probability of detecting this signal at the output of a v-th order power law for a fixed false alarm probability.

The false alarm probability is defined as

$$\alpha = 1 - F_Y(y_\alpha/H_0),\ y > 0$$

where y_α is the threshold (value of y) where the desired false alarm probability is maintained. For this case we find

$$y_\alpha^{1/v} = \sigma_n\ \Phi^{-1}[1 - \alpha/2].$$

The probability of detection (Pd) is therefore

$$\begin{aligned} Pd &= 1 - F_Y(y_\alpha/H_1) \\ &= 2\left\{1 - \Phi[\Phi^{-1}(1-\alpha/2)/(1+SNR)^{1/2}]\right\},\ y > 0 \end{aligned} \tag{29}$$

where $SNR = \sigma_s^2/\sigma_n^2$.

Once α is fixed for each v the probability of detection is independent of v and only depends on SNR. Therefore, the performance of all even order power laws will be the same assuming Gaussian processes based on the probability of detection criterion.

Case (II) V ODD

The odd order power law is given by

$$y = x^{v},\ v = 2k-1,\ k=1,2,\dots\ .$$

For $y > 0$ and $x \geq 0$ the solutions of x are represented by $x = y^{1/v} \exp(jn2\pi/v)$.

But there is only one real solution, viz., $x = y^{1/v}$.

For $x < 0$ and $y < 0$, the odd order power law is given by

$$-|y| = -|x|^{v}, \quad \text{or} \quad y_1 = x_1^{v} \text{ under the conditions}$$

$$\left(\begin{array}{ll} 0 < y_1 < \infty, & 0 < x_1 < \infty \\ -\infty < y < 0, & -\infty < x < 0 \end{array}\right)$$

So, the probability density function at the output is given by

$$f_y(y) = [v\ y^{(v-1)/v}]^{-1}\ f_x(y^{1/v}),\quad y > 0$$

$$f_{y_1}(y_1) = [v\ y_1^{(v-1)/v}]^{-1}\ f_{x_1}(y_1^{1/v}),\quad y_1 > 0$$

Integrating we obtain

$$F_y(y) = \int_{-\infty}^{y} f_y(Z)dz = \int_{0}^{\infty} f_{y_1}(Z)dz + \int_{0}^{y} f_y(Z)dz$$

$$= 1/2 + \int_0^{y^{1/v}/\sigma} (2\pi)^{-1} \exp(-g^2/\alpha)\, dy, \text{ or}$$

$$F_y(y) = \Phi[y^{1/v}/\sigma],\ y \geq 0 \quad .$$

If we fix a threshold to the right ($y_\alpha > 0$) we find for α,

$$\alpha = 1 - F_y(y_\alpha/H_0) \text{ and}$$

$$y_\alpha^{1/v} = \sigma_n\ \Phi^{-1}(1-\alpha) \quad .$$

Under this threshold the probability of detection is

$$Pd = 1 - F_y(y_\alpha/H_1) = 1 - \Phi[\Phi^{-1}(1-\alpha)/\sqrt{1+SNR}]$$

which is also independent of v. As SNR approaches infinity, for y_α finite, we obtain, $\lim_{SNR\to\infty} Pd \gg 1/2$.

With one threshold this is the best we can do. But for two-sided distributions even-order power laws are usually employed. However, if we set two thresholds located at y_α and $-y_\alpha$ for $y_\alpha > 0$, then

$$\alpha = 2[1 - F_y(y_\alpha/H_0)], \text{ which gives the threshold value at,}$$

$$y_\alpha = \sigma_n\, \Phi^{-1}(1 - \alpha/2).$$

Then

$$Pd = 2\left\{ 1 - \Phi[\Phi^{-1}(1-\alpha/2)/(1+SNR)^{1/2}]\right\} ,$$

which is identical to the results for the even order power laws.

We conclude that for zero mean Gaussian processes there is no essential difference in performance for the v-th order power laws based on the probability of detection criterion, if two thresholds are set for the odd orders. These results also hold true if $L \neq 1$.

An interesting question presents itself. Can performance improvements be obtained by summing over a large number of samples (n large) compared with the single degree of freedom case? For randomly occurring signals the answer is not obvious, as would be so for purely Gaussian signals. By comparing equations (29) and (19) under the same SNR and false alarm probability we find that the FDK of case 1 can have higher asymptotic probability of detection levels compared with the probability of detection for the single degree of freedom case of equation (19), depending on the probability of occurrence (L) of the signal. This conclusion also holds for the FDK of cases 2 and 3.

The probability of detection cannot be obtained for the FDK of case 2 with one degree of freedom since the joint distribution of the numerator and denominator is unknown. This is also true for the FDK of case 1 if the degrees of freedom are greater than two but not large enough to assume Gaussianity. On the other hand, if $cov(x_1 x_{22})$ equals zero than with only a limited number of degrees of freedom the probability of detection can be obtained.

We will now give several examples comparing the FDK of cases 2 and 3 with the PSD. Figures 5 through 9 represent a comparison of the PSD with cases 2 and 3 based on the asymptotic probability of detection vs L for fixed SNR or vs SNR for fixed L. In all the figures the false alarm probability is 10^{-3} and the sample size (n) is 2000. The results show that the FDK of case 2 has a higher asymptotic probability of detection in all the figures even for a Gaussian process, i.e., L = 1. The last section gave reasons for this result and other comments are given in appendix A. Under some situations the FDK of case 3 has slightly higher asymptotic probability of detection compared with the PSD but only for non-Gaussian processes. Figures 8 and 9 clearly show how the APD changes for the FDK of case 3 and the PSD as L is varied from .1 to 1.0. For a non-Gaussian process (L = .1) the FDK of case 3 has higher APD levels vs SNR compared with the PSD. Whereas, the reverse is true for a Gaussian process (L = 1) as shown in figure 9. These conclusions do not change with changes in the sample size or false alarm probability.

In practice the asymptotic probability of detection for the FDK will be closer to the results of case 2 if the correlation between x_1 and x_{22} is high. On the other hand, if the correlation is zero or low then the FDK of case 3 should be utilized instead of normalizing the fourth-order estimate by an uncorrelated noise sample. This is true because the uncorrelated noise sample tends to increase the variance and, therefore, reduce the APD.

IV. SIMULATIONS

Several simulations of the ideas presented in the previous sections will be given to check the theoretical results. We will utilize equation (12) as

the data model but here both real and imaginary parts will be included in the simulations. Therefore, the real and imaginary parts, respectively, are as follows:

$$X_R(q) = N_R(1) + m_R(q)S_R(q)$$
$$X_I(q) = N_I(q) + m_I(q)S_I(q) \quad . \tag{27}$$

The SNR is defined as, SNR = $E[S_R(q)^2]/E[N_R(q)^2] = E[S_I(q)^2]/E[N_I(q)^2]$, since both real and imaginery parts will have identical statistics in the following simulations. The power spectrum density is, therefore,

$$P(F_p) = (1/n)\sum_{q=1}^{n}[X_R(q)^2 + X_I(q)^2] \quad , \tag{28}$$

and the corresponding frequency domain kurtosis estimate will be defined in the following way,

$$K_z(F_p) = [K_z^R(F_p)^2 + K_z^I(F_p)^2]^{1/2} \quad . \tag{29}$$

Other estimates could be defined for the FDK depending on the intended application. But equation (29) will be our basis for comparison with equation (28) in the simulations.

A. Simulation of Case 1

The real data example of figure (4) has already demonstrated the usefulness of the FSK for case 1. Therefore, only limited examples will be

presented. The parameters needed to define a particular simulation will be denoted in the form (z, L, SNR, n), where z equals 1 or 2 depending on whether the FDK if case 1 or case 2 is being considered. The FDK of case 3 will not be simulated here.

Consider the simulation (1, .005, 16.0, 200) shown in figure 10. The data were generated using a 1024 point FFT. The top graph represents the PSD expressed in equation (28). The lower graph is the corresponding FDK estimate of equation (29). The signal only occurs once for each of 200 frequency locations starting at frequency 100 during the 200 consecutive FFT data samples. But the temporal location is random in each frequency and therefore unknown. The resultant estimates have a form of a broadband signal. From the raw data in the figure we see that the FDK estimate identifies the frequency location of the randomly occurring signal but this information is not present in the PSD estimate. This result was predicted in figure (3). So, the simulation corroborates the theoretical results for the FDK of case 1.

Another possible application of these results is the identification of the track of a signal that is changing its frequency with time. The instantaneous frequency location could not be identified from the FDK but it could possibly be deduced from the results. The usefulness of the FDK in the application would have to be carefully compared with tracking methods. Since this would require a detailed study it has not been treated. But it does appear that the FDK would be easier to implement. So a study is probably warranted in the future. These considerations would also apply for spatially tracked signals.

B. Simulation of Case 2

In the following simulations we shall perform a detection experiment. For noise only a threshold will be found for the PSD and FDK by picking the highest level in each case of the 512 frequency locations. Then for signal and noise present the number of frequency locations in each case exceeding their corresponding thresholds will be counted. Starting at location 100 there will be 100 locations with a randomly occurring signal present in this simulation. Since the FDK of case 2 requires a noise-only second-order estimate to normalize the fourth-order estimate of signal and noise, these results may be of theoretical interests rather than having practical value. On the other hand, if a noise-only second-order estimate is available which was highly correlated with the noise in the fourth-order estimate of signal and noise then these results would be of practical value. It should also be pointed out that only a noise-only second-order estimate is needed and not knowledge of any particular noise sample.

Figures 11, 12, and 13 represent the results of a simulation of (2, .02, 4,200), (2, .04, 4,200), and (2, .08, 4,200). As before the PSD estimate is in the top graph and its corresponding FDK estimate is in the lower graph. The number of locations exceeding the threshold are 1, 6, and 34 for the PSD and 47,88, and 100 exceeded the threshold for the FDK in figures 11, 12, and 13, respectively.

The theory also predicts that the FDK of case 2 will have a higher probability of detection compared with the PSD estimate even for Gaussian processes. This result was checked in the following simulation. The results

of the simulation are shown in figure 14, under the conditions (2, 1, SNR, 200) where SNR equals .1, .16, .25, and .4. The respective detection results are 6, 13, 45, and 97 for the PSD estimate, and 52, 69, 99, and 100 for the FDK estimate. These results obviously support the theoretical predictions for knowledge of a noise-only second-order estimate. Appendix A points out that if the PSD estimate is also normalized by a noise-only second-order estimate its probability of detection would appear to be better than the FDK's but the false alarm rate cannot be maintained at a desired low level.

V. SUMMARY

Since signals often appear to occur randomly, especially in underwater acoustic detection problems, and the corresponding detector based on the likelihood ratio has a complex structure, fourth-order and kurtosis estimates were considered as alternative processing methods. These higher-order methods were compared with second-order estimates for signals which occur randomly in time and frequency and which could be described as a non-Gaussian process. It was initially believed that a second-order estimate was optimum for Guassian processes. However, it was shown theoretically, based on the asymptotic probability of detection that if a noise-only second-order estimate was available to normalize a fourth-order estimate of signal and noise then the resultant modified kurtosis estimate had higher probability of detection levels even for Gaussian processes. On the other hand, if an independent noise sample is used to normalize a second-order or fourth-order estimate the overall performance based on the asymptotic probability of detection will be degraded compared with unnormalized second-order or fourth-order estimates, respectively. Only if there is positive covariance between the normalizing noise sample and the second-order or fourth-order estimate can performance be improved. This result could impact current sonar processing methods. Simulations were presented which support these results.

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APPENDIX A

SECOND-ORDER ESTIMATES

Let the power spectrum density estimate of equation (14) be normalized by a noise-only second-order estimate as follows:

$$P(F_p) = x_1/x_2 = \left\{(1/n)\sum_{q=1}^{n} [x(q)/H_1]^2\right\}/\left\{(1/n)\sum_{q=1}^{n} [x(q)/H_0]^2\right\} \qquad (A1)$$

Asymptotically, $P(F_p)$ will also converge to a Gaussian process, under H_1, according to Cramer's [10] convergence proof.

Employing the data model of equation (12) we obtain the expected value of equation (A1) as

$$E[P(F_p)] = 1 + L\ SNR.$$

The components of the variance of equation (A1) are as follows:

$$VAR[X_1] = (1/n)\ [2N^2 + 4L\ SN + (3L - L^2)\ S^2\ ]$$

$$VAR(X_2] = (1/n)\ [2N^2]$$

$$F_1 = 1/N$$

$$F_2 = -\ (1/N)(1 + L\ SNR)$$

$$COV[X_1\ X_2] = (1/n)\ 2N^2$$

Therefore, the variance of equation (A1) is given by

$$VAR[P(F_p)] = (1/n)\ a_8 \quad ,$$

where, $a_8 = (3L + L^2)\,SNR^2 + 4L\,SNR$.

As SNR approaches zero $VAR[P(F_p)]$ also approaches zero. This result makes sense, because when the SNR equals zero $P(F_p)$ equals one, and therefore its variance would equal zero.

From equation (13) we obtain the result

$$APD = 1 - \Phi[-\,L\,SNR\,n/\,a_8] \quad .$$

As SNR approaches zero the APD approaches .5. However, to avoid complications in the limit we will assume that SNR approaches zero but does not exactly reach it under H_0. This can be stated more precisely as follows: As $H_1 \to H_0$, $SNR \to \delta$, for $\delta << 1$, and $APD \to .5 + D(\delta)$, where $D(\delta) \to 0$ as $\delta \to 0$. This result also establishes the limiting false alarm probability of .5 for equation (A1). Therefore, the false alarm probability cannot be controlled at a desired low level. On the other hand, the APD would have higher levels compared with the FDK of case 2 as a function of SNR. This may not be of interest, however, due to the high false alarm probability.

If we now let the PSD estimate of equation (14) be normalized by an uncorrelated noise-only second-order estimate, say from an adjacent frequency location, which would give $COV(X_1, X_2) = 0$, then the mean and variance would be as follows:

$$E[P(F_p)] = 1 + L\,SNR$$

$$VAR[P(F_p)] = (4/n)\,Q_3,$$

where, $Q_3 = [1 + 2\ L\ SNR + (3L + L^2)\ SNR^2/4]$.

These results give an APD of the form

$$APD = 1 - \Phi\left\{[\Phi^{-1}(1 - \alpha) - L\ SNR\sqrt{n/2}/\sqrt{2}]/Q_3^{1/2}\right\} . \tag{A2}$$

The false alarm probability can now be maintained at a desired level. But, for small SNR levels, i.e., $SNR^2 << 1$, the APD of equation (A2) will be lower than the corresponding APD levels of equation (15) for the same false alarm probability. This is true because of the factor $\sqrt{n/2}$ in equation (A2). If L equals one, then the APD of equation (15) will always be higher independently of SNR. For high SNR there may be values of L where the APD of equation (A2) is higher but the factor $\sqrt{n/2}$ must be overcome before this is true.

APPENDIX B

VARIANCE OF THE FDK ESTIMATE OF CASE 1

Here we present the major points for the derivation of the variance based on Cramer's method [10] for the following equation

$$K_1(F_p) = X_1/X_{21} \quad ,$$

where $X_1 = (1/n)\sum_{q=1}^{n} X(q)^4$, and, $X_{21} = [(1/n)\sum_{q=1}^{n} X(q)^2]^2$.

From section II the form of the variance is known and it can be expressed as,

$$VAR[K_1(F_p)] = VAR(X_1)F_1^2 + VAR(X_{21})F_2^2 + 2F_1F_2\ COV[X_1X_{21}] \quad . \tag{B1}$$

The covariance plays an important role in minimizing the total variance of equation (B1).

The partial derivatives F_1 and F_2 of equation (B1) are as follows:

$$F_1 = \partial K_1(F_p)/\partial X_1 \Bigg|_{\substack{X_1 = \overline{X}_1 \\ X_{21} = \overline{X}_{21}}} = 1/\left\{(1/n)[\overline{X(q)^4} - (\overline{X(q)^2})^2] + (\overline{X(q)^2})^2\right\}$$

where the overbar, when used, will represent the expected value,

$$F_2 = \partial K_1(F_p)/\partial X_{21} \Bigg|_{\substack{X_1 = \overline{X}_1 \\ X_{21} = \overline{X}_{21}}} = -\overline{X(q)^4}/\left\{(1/n)\ [\overline{X(q)^4} - (\overline{X(q)^2})^2 + (\overline{X(q)^2})^2\right\}^2$$

The terms $VAR(X_1)$ and $VAR(X_{21})$ can be expressed in the form

$$VAR(X_1) = (1/n)\ [\overline{X(q)^8} - (\overline{X(q)^4})^2] \quad ,$$

$$VAR(X_{21}) = (4/n)\ [\overline{X(q)^4} - (\overline{X(q)^2})^2 - (X(q)^2)^4] \quad ,$$

if the terms which converge to zero faster than $(1/n)$, i.e., $O(1/n^2)$, are neglected.

The covariance is defined as

$$COV[X_1 X_{21}] = E(X_1 X_{21}) - E(X_1)\ E(X_{21}) \quad .$$

If X_1 and X_{21} are independent, i.e., $E(X_1 X_{21}) = E(X_1)\ E(X_{21})$ then $COV(X_1 X_{21}) = 0$.

From the definition of X_1 and X_{21} we obtain

$$COV(X_1 X_{21}/H_1) = E\left\{(1/n)\sum_{q=1}^{n} X(q)^4 [(1/n)\sum_{q=1}^{n} X(q)^2]^2\right\}$$

$$- E[(1/n)\sum_{q=1}^{n} X(q)^4] E\left\{[(1/n)\sum_{q=1}^{n} X(q)^2]^2\right\} \quad ,$$

where we have indicated that the denominator contains signal and noise by H_1 in the covariance expression. We do this to distinguish between case 1 and case 2 of the FDK which contains only noise in the denominator. By neglecting terms of order $O(1/n^2)$ we obtain

$$COV(X_1X_{21}/H_1) = (2/n)\left\{ \overline{X(q)^6}\ \overline{X(q)^2} - \overline{X(q)^4}\ (\overline{X(q)^2})^2 \right\}.$$

Substituting all the terms into equatin (B1) we obtain

$$VAR[K_1(F_p)] = (1/n)\left\{ [\overline{X(q)^8} - (\overline{X(q)^4})^2][\overline{X(q)^2}]^{-4} \right.$$

$$+ 4[\overline{X(q)^4}\ (\overline{X(q)^2})^2 - (\overline{X(q)^2})^4](\overline{X(q)^4})^2\ (\overline{X(q)^2})^{-8} \tag{B2}$$

$$\left. - 4[\overline{X(q)^6}\ \overline{X(q)^2} - \overline{X(q)^4}\ (\overline{X(q)^2})^2]\ \overline{X(q)^4}(\overline{X(q)^2})^{-6} \right\}$$

In order to express equation (B2) in the desired parameterized form the data model of equation (12) is substituted into each term. But instead of doing this for each term, only the general term of equation (B2) will be evaluated. The other terms will follow in a similar way. Therefore, we shall express $VAR(X_1)$ in parameterized form. The most general term is

$$VAR(X_1) = \overline{X(q)^8} - (\overline{X(q)^4})^2 \tag{B3}$$

From equation (7) the first term of equation (B3) is

$$\overline{X(q)^8} = (1/M^4)\ OP_8\left\{ E\left(X(i_1)\ X(i_2)\ X(i_3)\ X(i_4)\ X(i_5)\ X(i_6)\ X(i_7)\ X(i_8) \right) \right\}$$

where we have defined the operator

$$OP_8\left\{\quad\right\} = \sum_{i_1=0}^{M-1}\sum_{i_2=0}^{M-1}\sum_{i_3=0}^{M-1}\sum_{i_4=0}^{M-1}\sum_{i_5=0}^{M-1}\sum_{i_6=0}^{M-1}\sum_{i_7=0}^{M-1}\sum_{i_8=0}^{M-1}\left\{\quad\right\} COS(F_p i_1)\ cos(F_p i_2)$$

$$cos(F_p i_3)\ cos(F_p i_4)\ cos(F_p i_5)\ cos(F_p i_6)\ cos(F_p i_7)\ cos(F_p i_8)$$

and where each $X(i)$ is also a function of q, i.e., $X(i,q) = N(i,q) + M(q)S(i,q)$, but there is no need to express that fact in the equation.

Since $N(i,q)$ and $S(i,q)$ are Gaussian $OP_8\{\ \}$ can be expanded as follows:

$$(1/M^4)\ OP_8\left\{ E[X(i_1)\ X(i_2)\ X(i_3)\ X(i_4)\ X(i_5)\ X(i_6)\ X(i_7)\ X(i_8)\right\}$$

$$= 105\ N^4 + L\ 420\ S\ N^3 + 630\ L\ S^2\ N^2 + L\ 420\ N\ S^3 + L\ 105\ S^4$$

which is not a function of the transform size M. By defining $OP_4\{\ \}$ for the second term and proceeding in a similar way equation (B3) reduces to

$$\overline{X(q)^8} - (\overline{X(q)^4})^2 = 96\ N^4 + 384\ L\ S\ N^3 + (612L - 36L^2)\ S^2\ N^2$$
$$+ (420L - 36L^2)\ S^3\ SN + (105L - 9L^2)\ S^4$$

The other terms are obtained similarly. These results are then substituted into equation (B2) to obtain equation (17), which is also not a function of the transform size M.

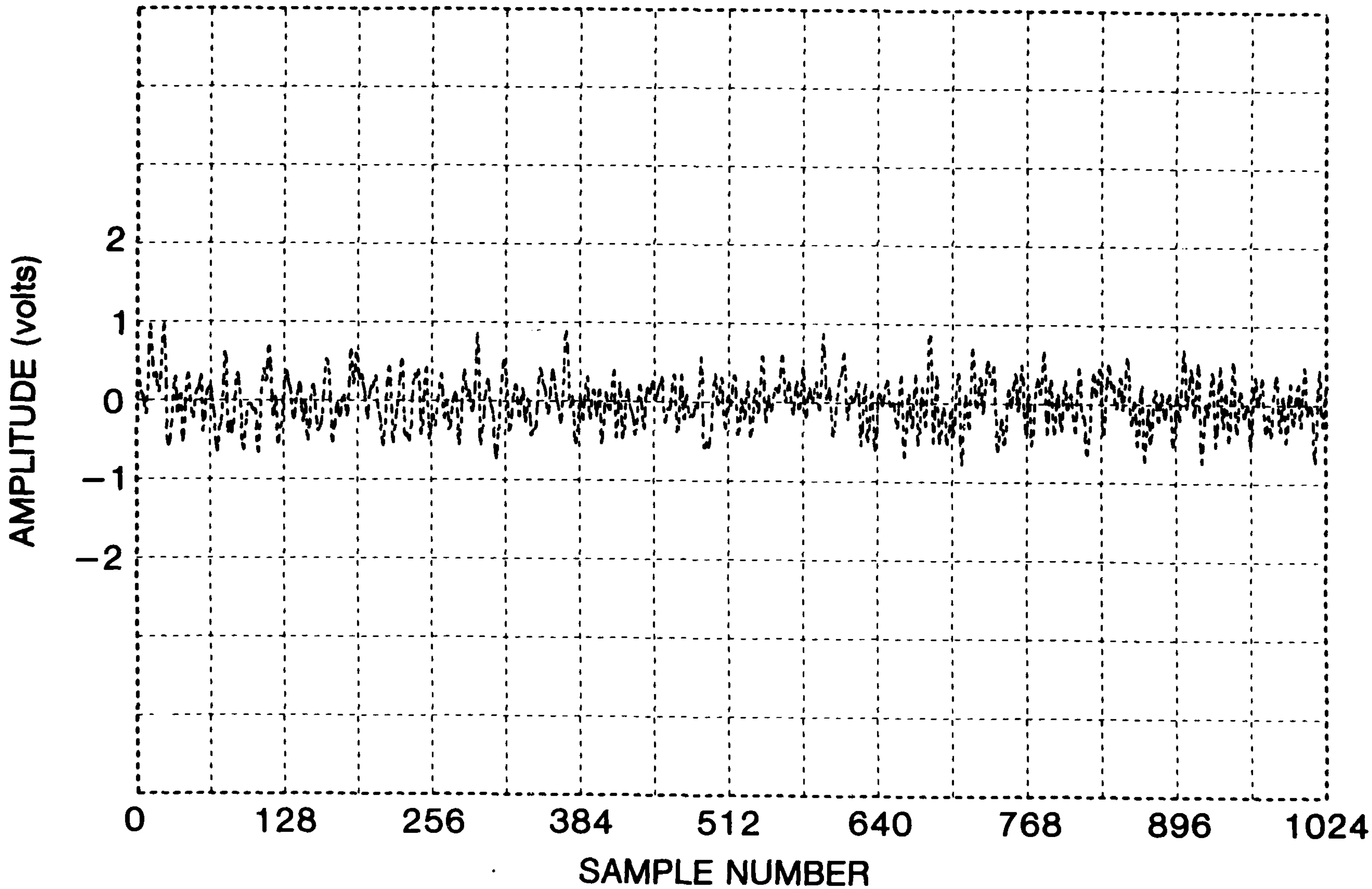

Figure 1. Ambient Noise

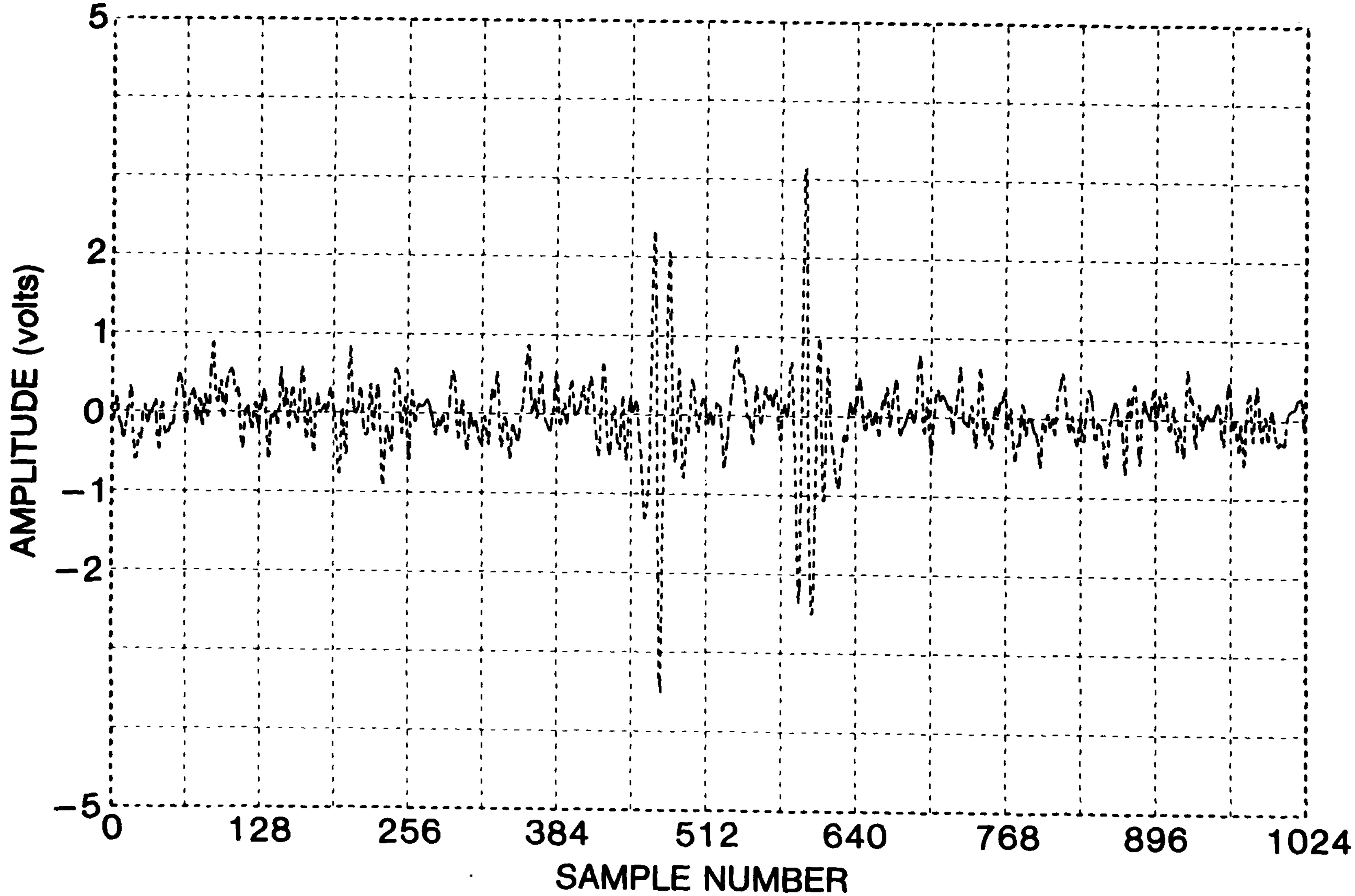

Figure 2. Ambient Noise with randomly occurring signal

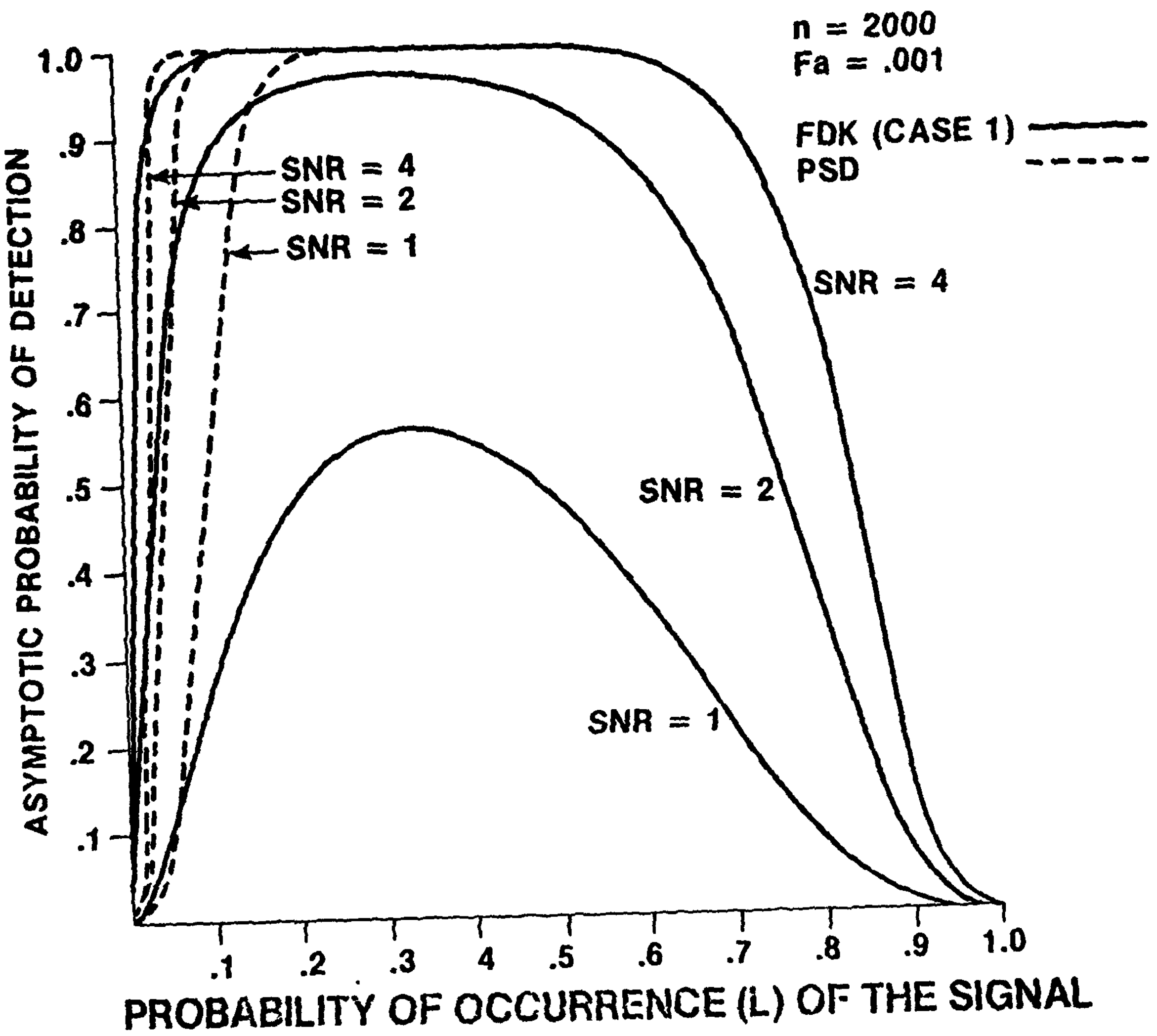

Figure 3. Asymptotic Probability of Detection vs Probability of Occurrence (L) of the signal. FDK (Case 1) and PSD.

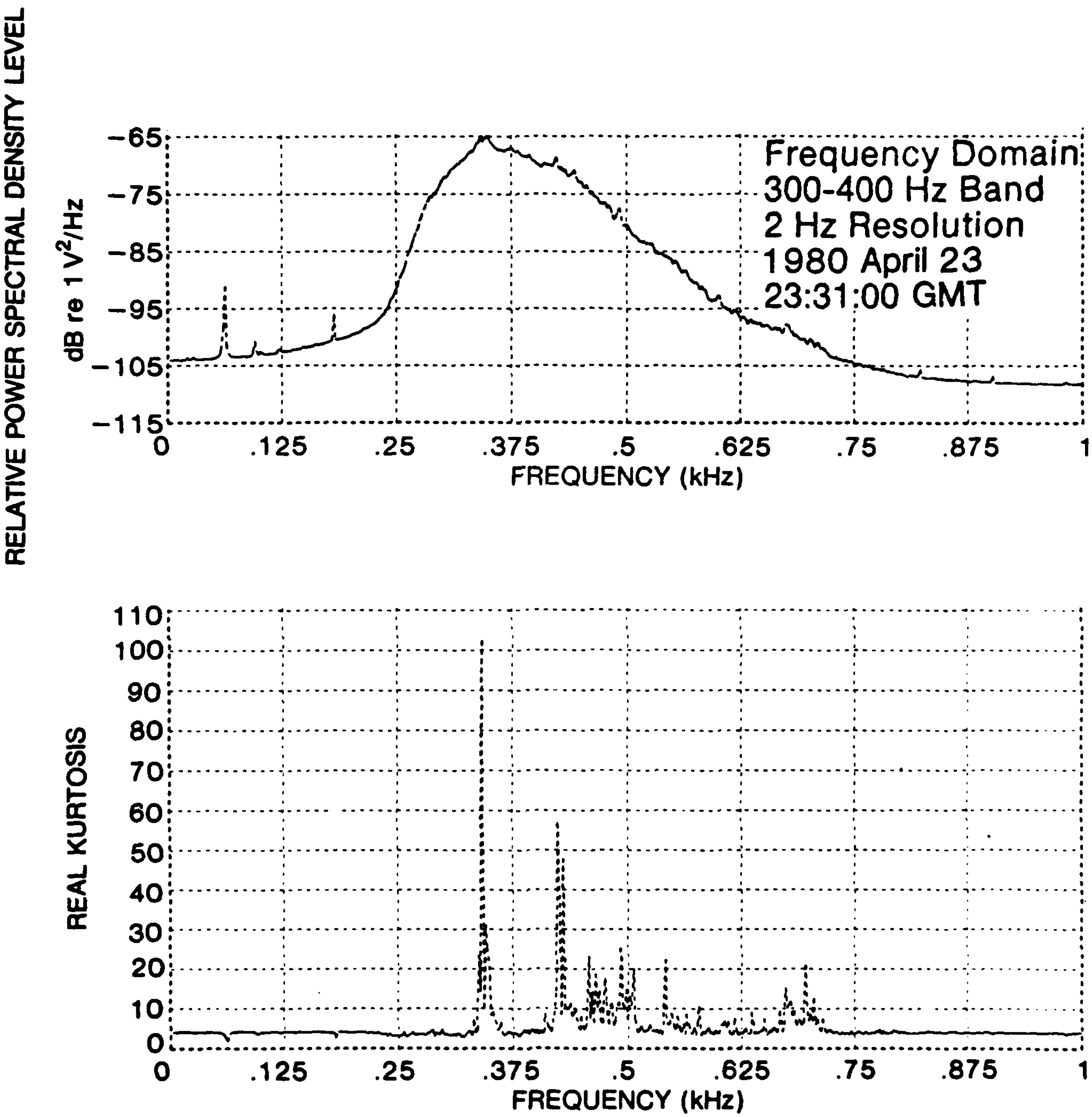

Figure 4. Real Arctic Data: Top - Power Spectrum Density Estimate; Bottom - Frequency Domain Kurtosis Estimate

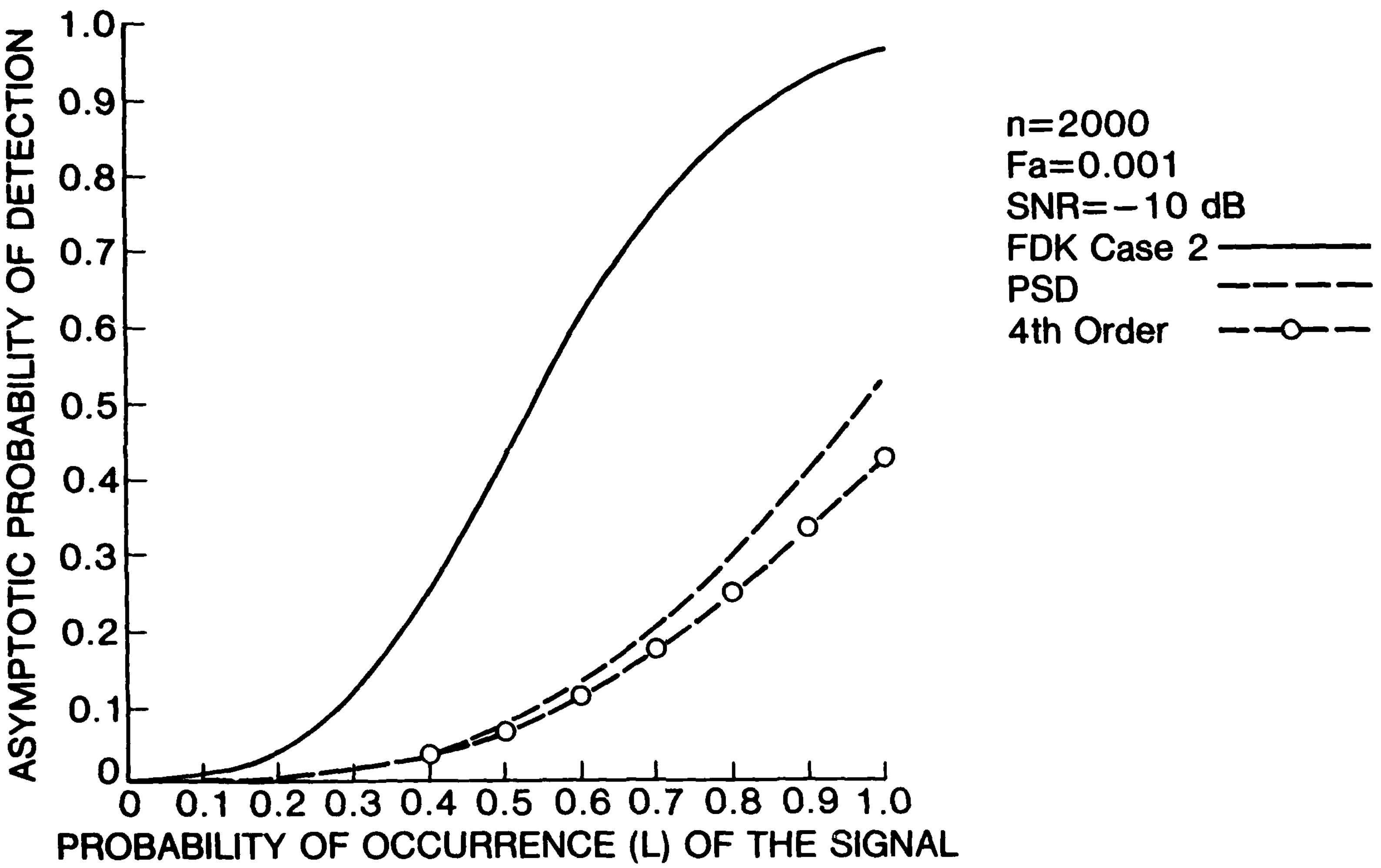

Figure 5. Asymptotic Probability of Detection vs Probability of Occurrence (L) of the Signal. FDK (Case 2), PSD, and Fourth-order. SNR = -10 dB.

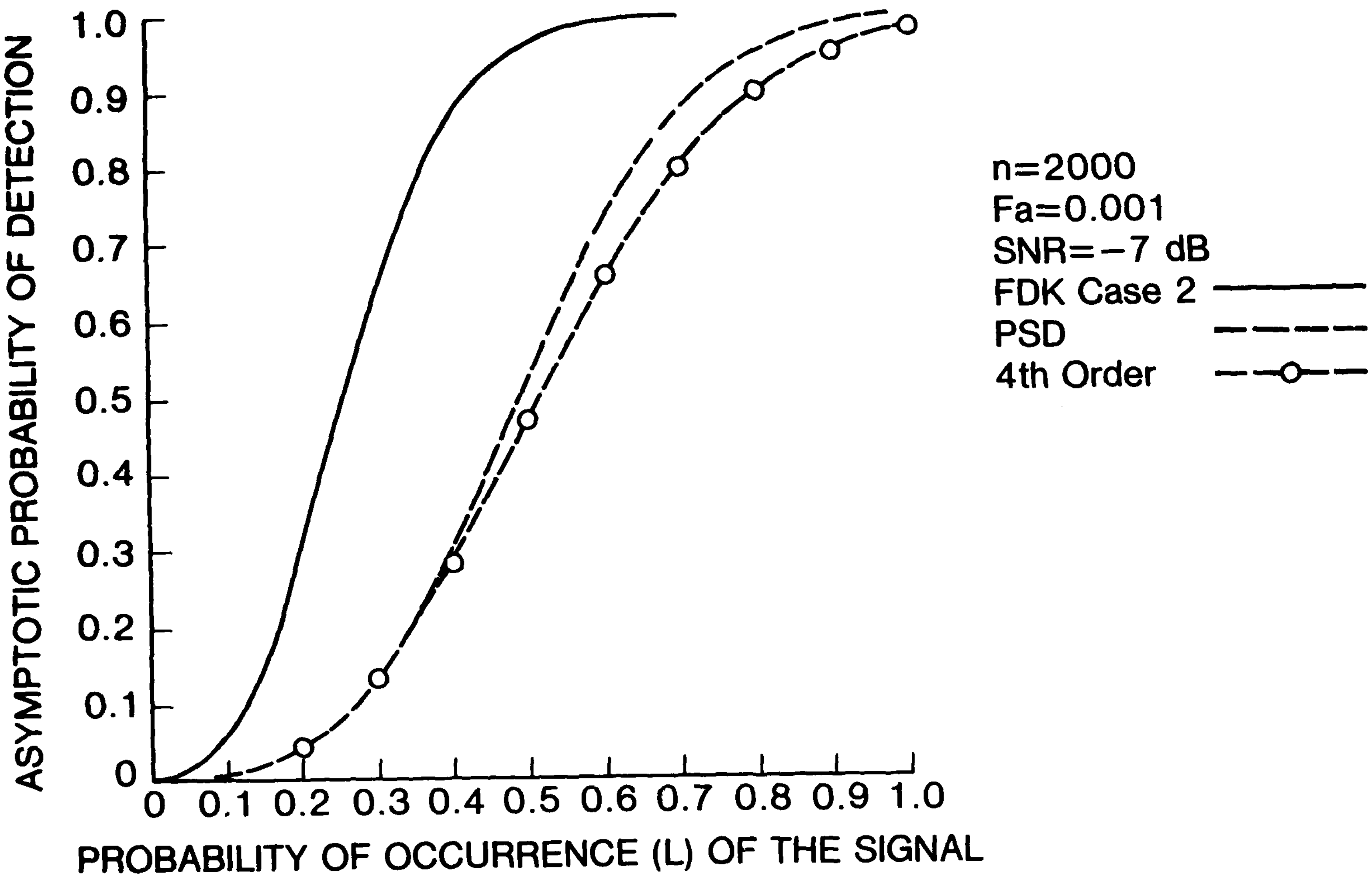

Figure 6. Asymptotic Probability of Detection vs Probability of Occurrence (L) of the Signal. FDK (Case 2), PSD, and Fourth-order. SNR = -7 dB.

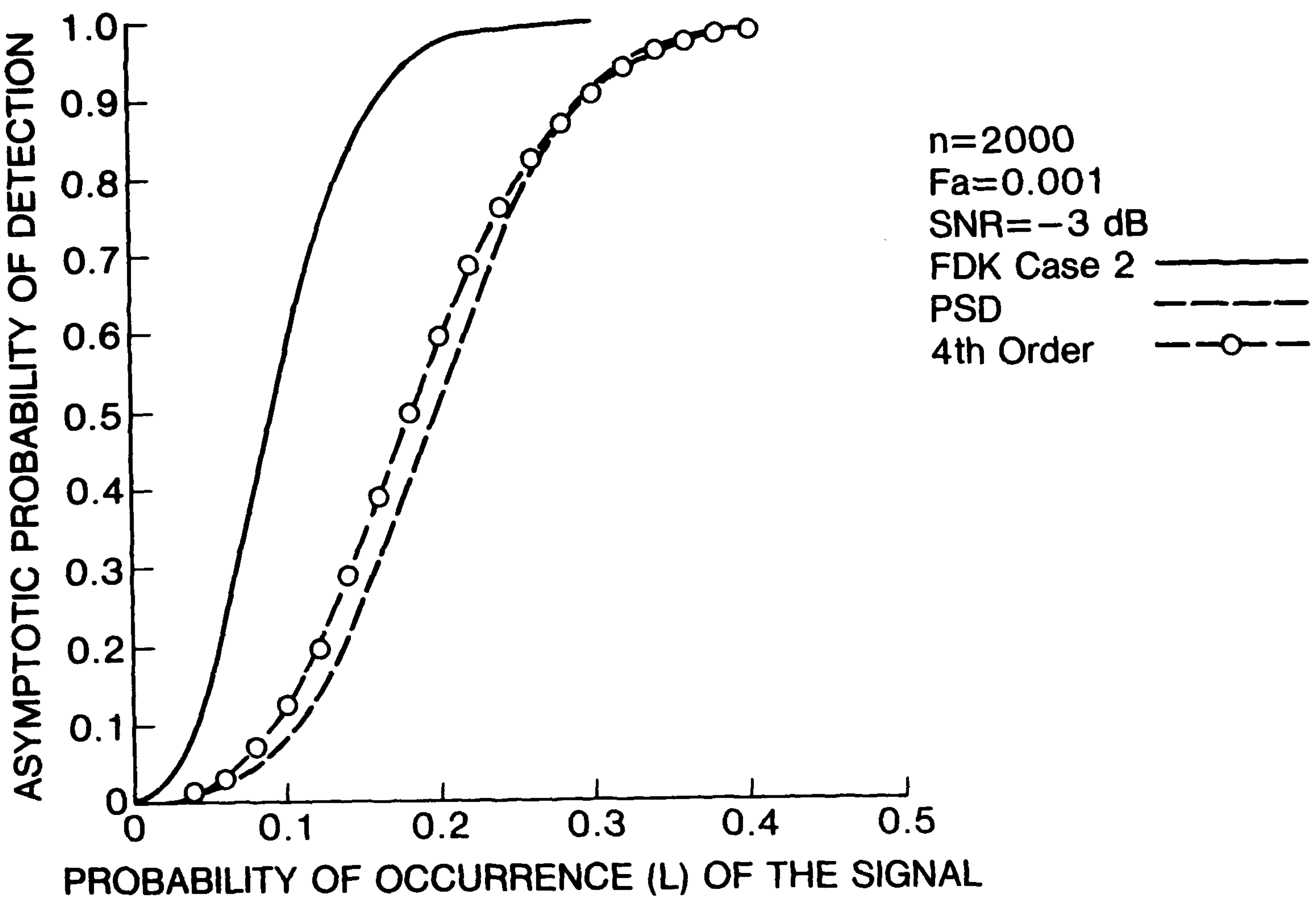

Figure 7. Asymptotic Probability of Detection vs Probability of Occurrence (L) of the signal. FDK (Case 2), PSD, and Fourth-order. SNR = -3 dB.

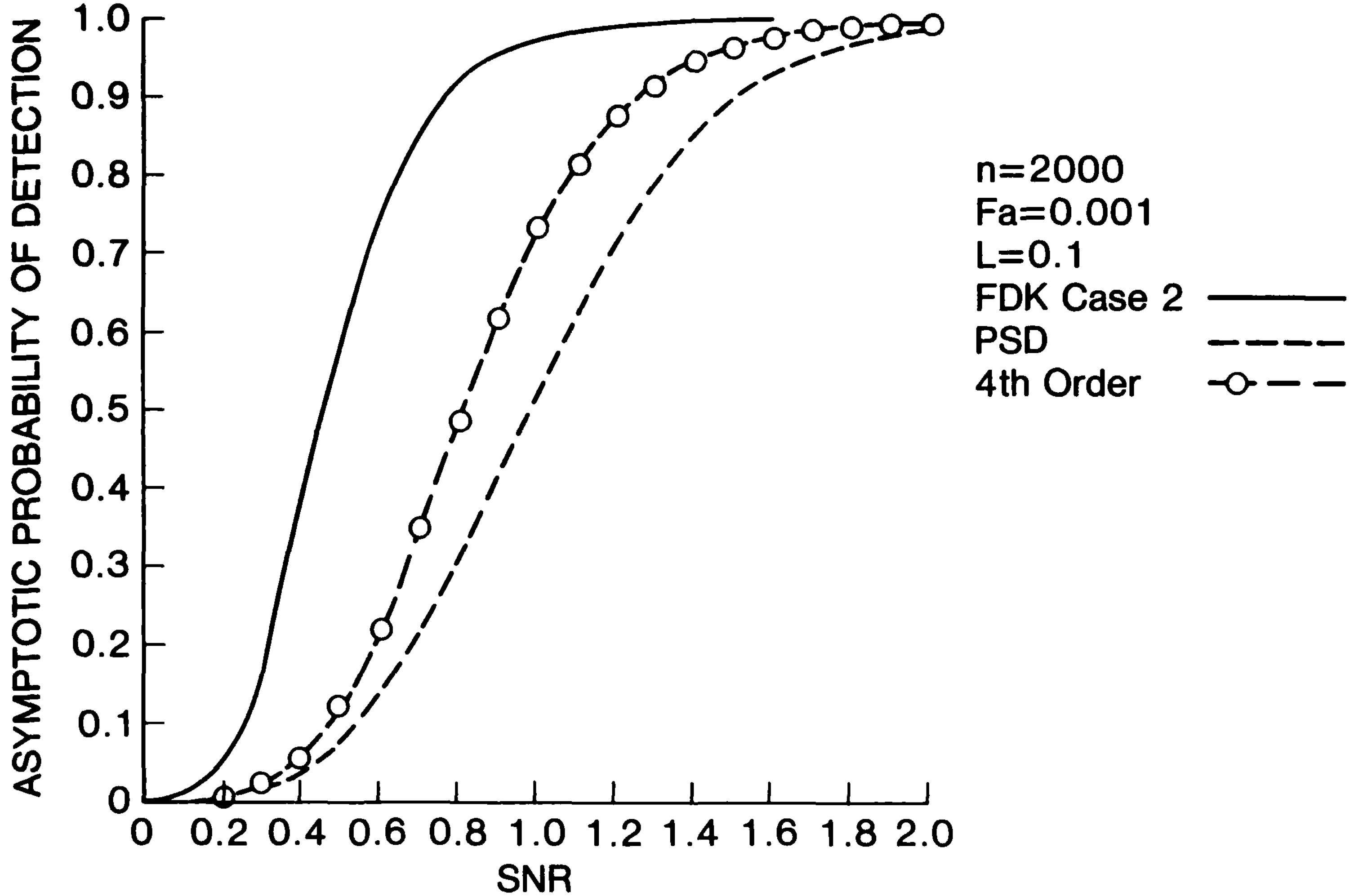

Figure 8. Asymptotic Probability of Detection vs Signal-to-Noise ratio. FDK (Case 2), PSD, and Fourth-order. L = .1.

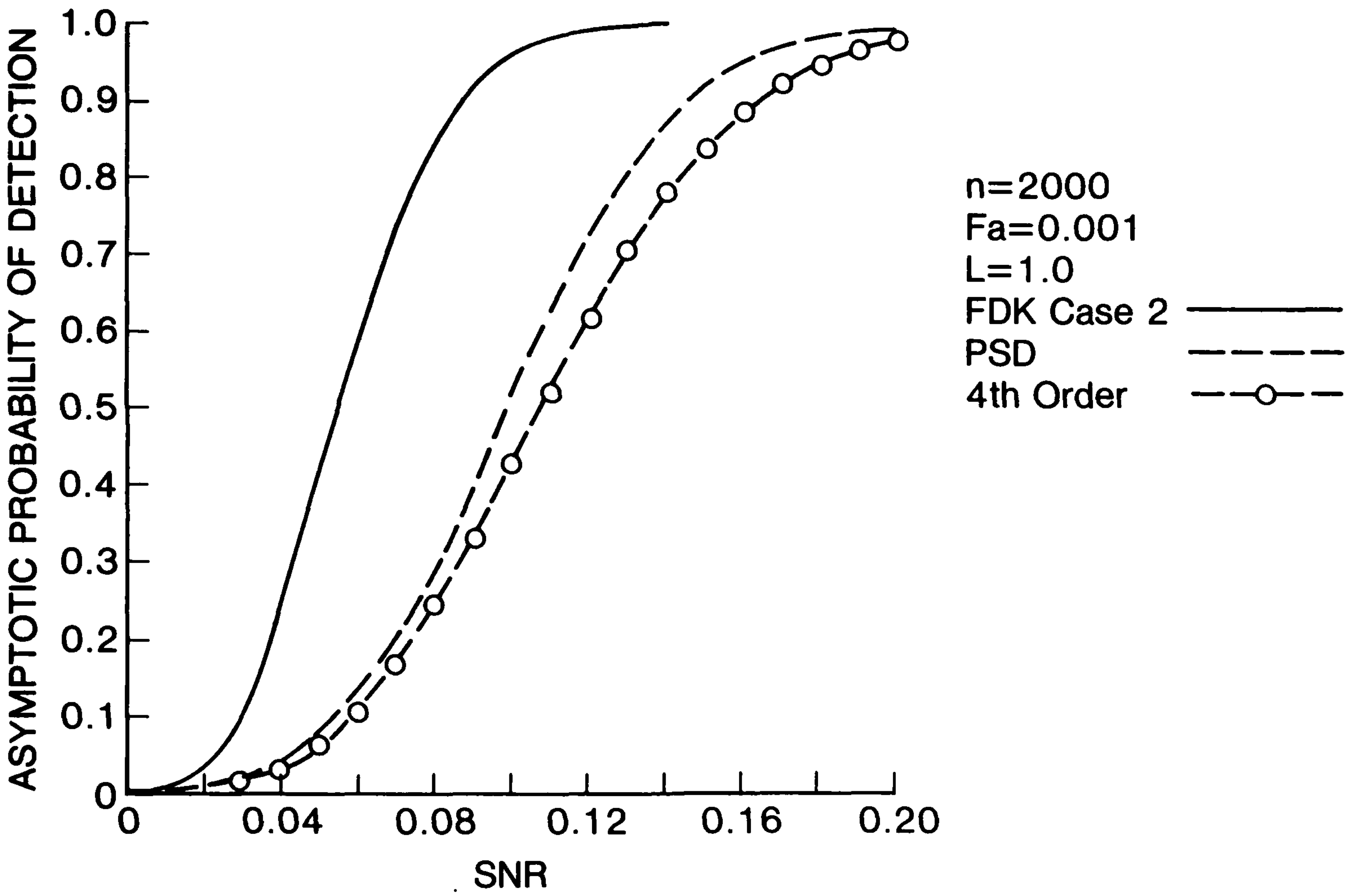

Figure 9. Asymptotic Probability of Detection vs Signal-to-Noise ratio.

L = 1.0.

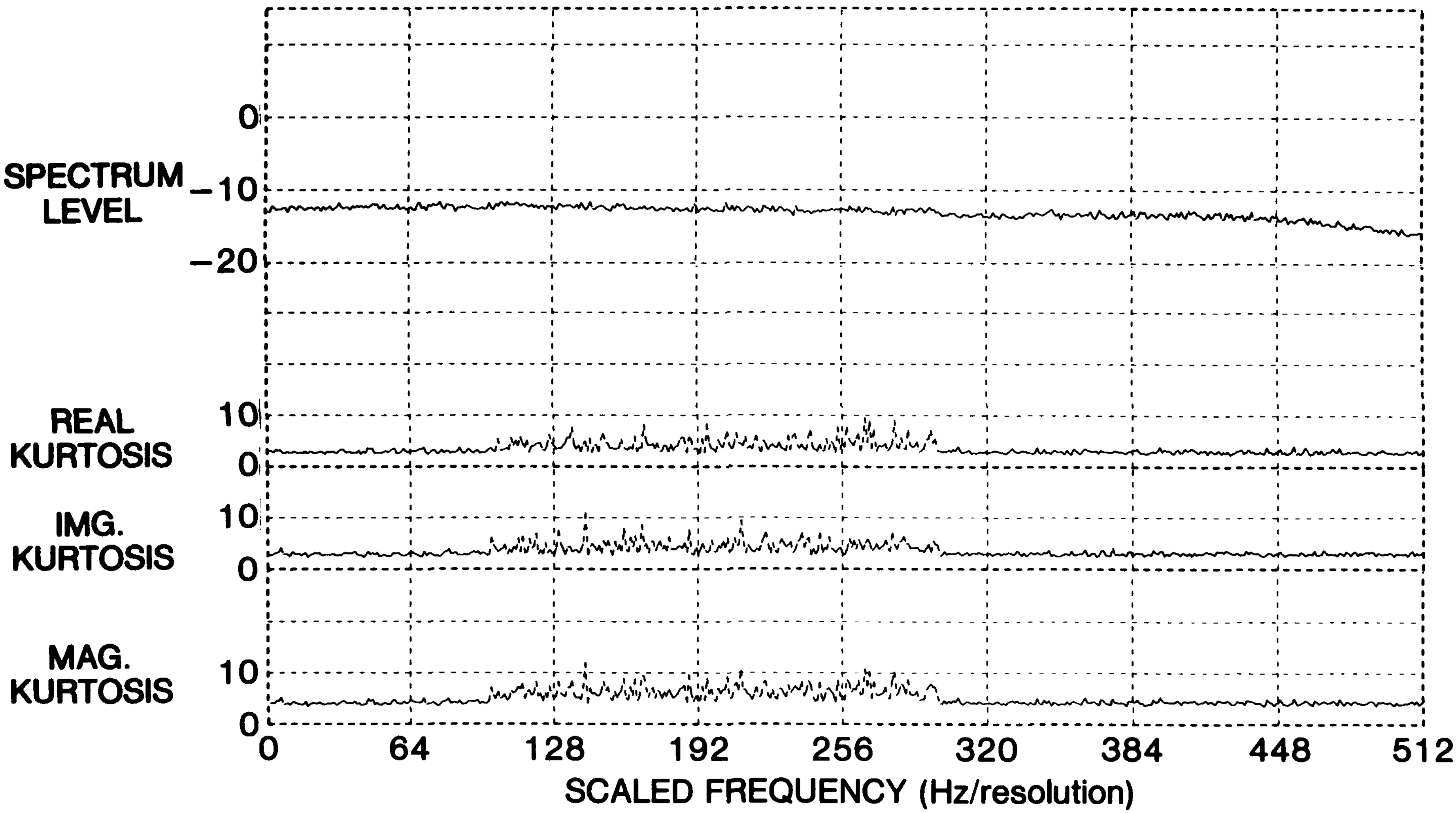

Figure 10. Simulation of FDK of Case 1. PSD, Real Kurtosis, Imaginary Kurtosis, and Magnitude of Kurtosis.

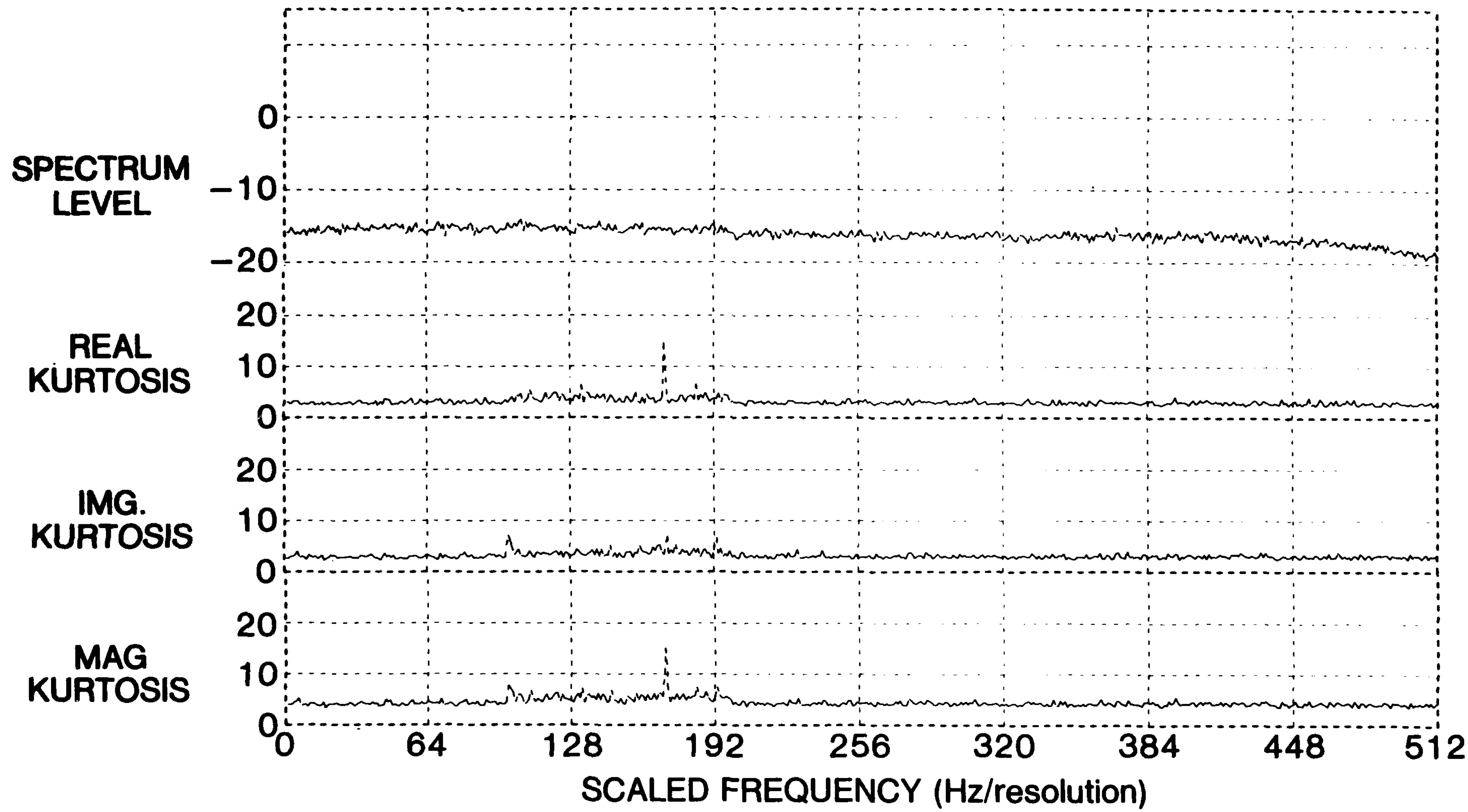

Figure 11. Simulation of FDK of Case 2. PSD, Real Kurtosis, Imaginary Kurtosis, and Magnitude of Kurtosis. L = .02.

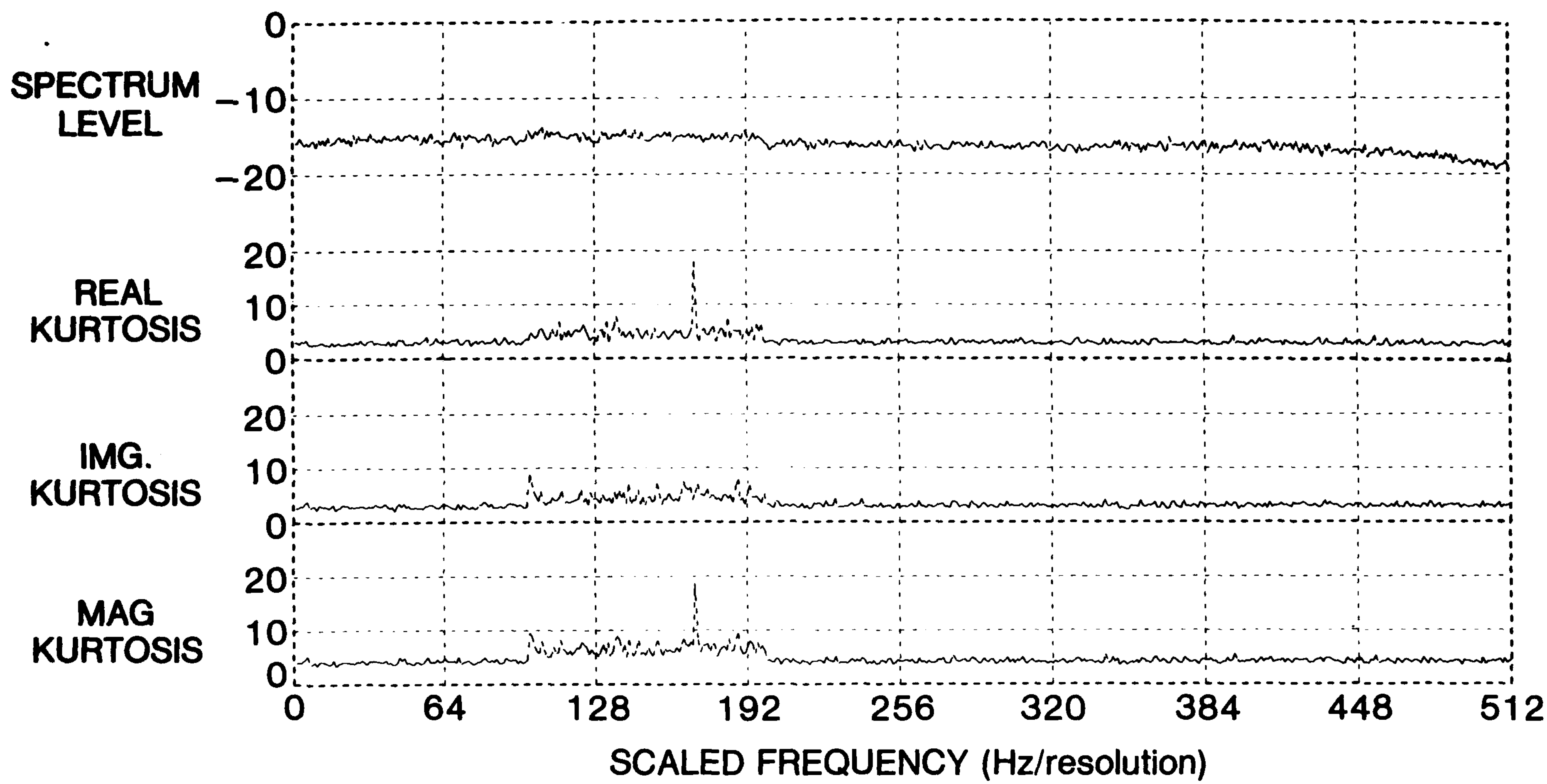

Figure 12. Simulation of FDK of Case 2. PSD, Real Kurtosis, Imaginary Kurtosis, and Magnitude of Kurtosis. L = .04.

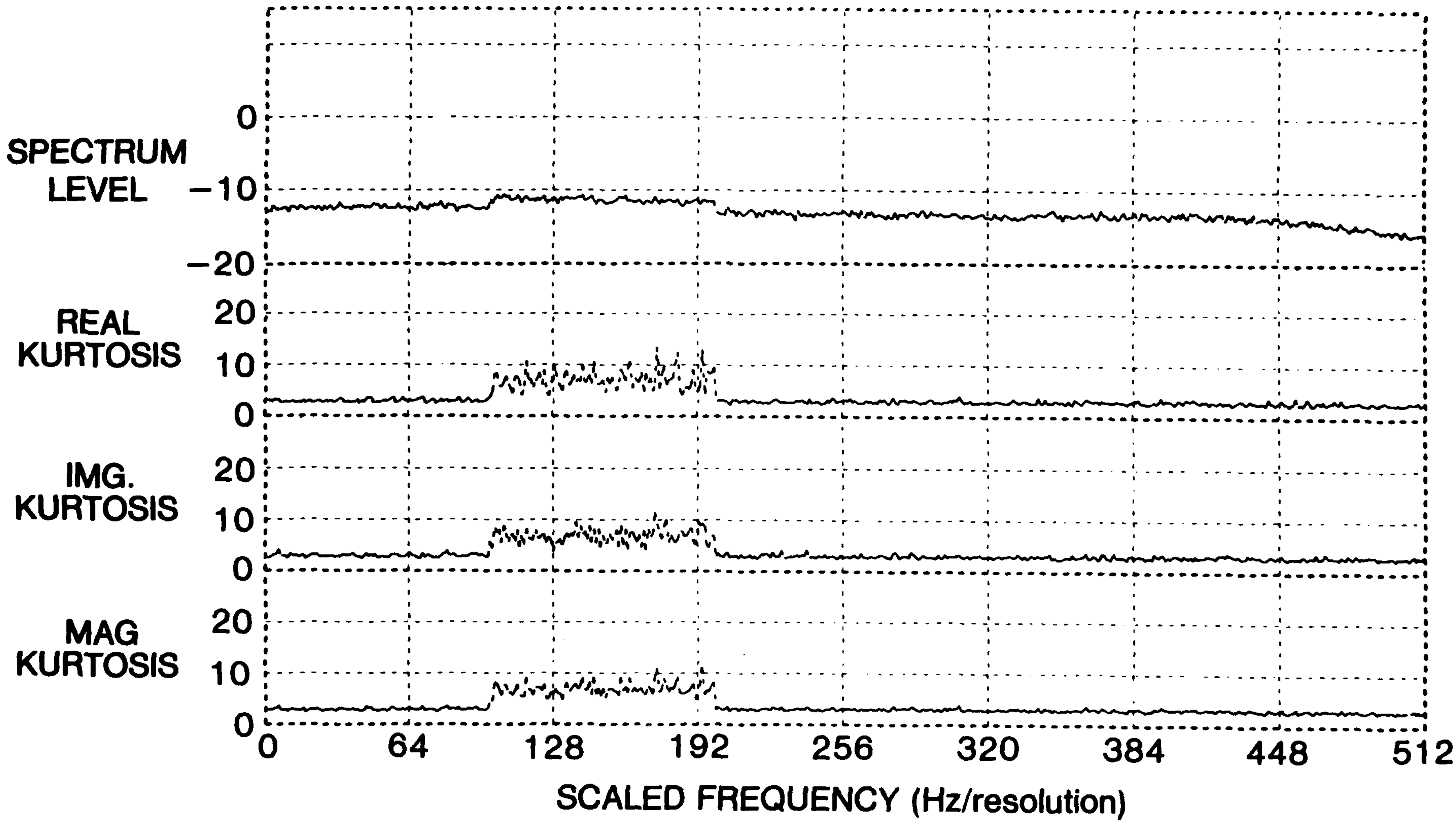

Figure 13. Simulation of FDK of Case 2. PSD, Real Kurtosis, Imaginary Kurtosis, and Magnitude of Kurtosis. L = .08.

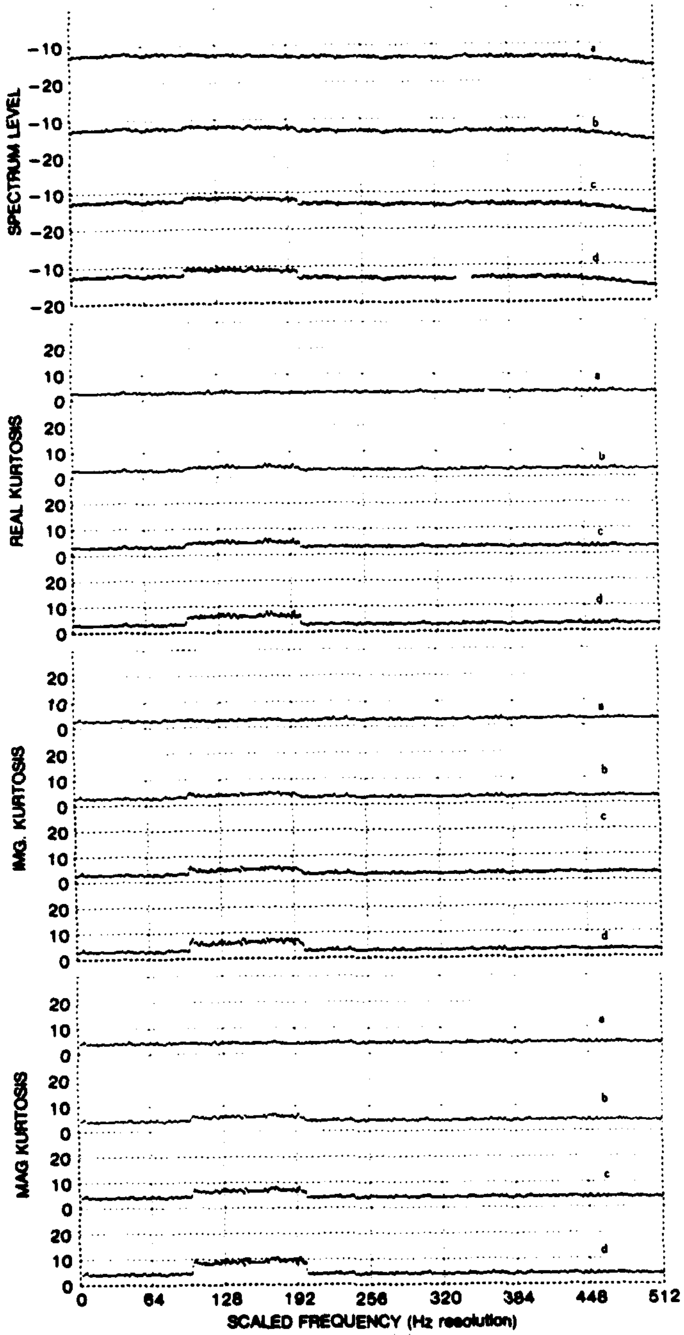

Figure 14. Simulation of FDK of Case 2 for Gaussian Process. PSD, Real Kurtosis, Imaginary Kurtosis, and Magnitude of Kurtosis. (a) SNR = .1, (b) SNR = .16, (c) SNR = .25 and (d) SNR = .4.

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Spectra and Frequency Domain Kurtosis Estimates
Roger F. Dwyer
Surface Ship Sonar Department
TM 841057
23 March 1984

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